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STAGGERED SOLUTION SCHEMES FOR DAM–RESERVOIR INTERACTION

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Traditionally, the dynamic reservoir effects in the dam–reservoir system are approximated using the added mass approach. The objective of this study is to develop unconditionally stable solutions for the dam-reservoir interaction problem in the time domain. The dam–reservoir system can be categorized as a coupled field system, in which two physical systems of fluid and structure interact only at the two domains interface. Two methods of staggered solution procedure are proposed for the dam–reservoir interaction. The first method, the staggered displacement method, is based on the approximation of the displacement from the structure's equation of motion. The second method, is based on the approximation of pressure from the fluid's equation of motion. Both methods are shown to be unconditionally stable when the two differential equations of the fluid and structure include damping terms. The staggered pressure method was modified for use when the equation of motion includes a diagonal mass matrix. Two different configurations of concrete gravity dams are used to investigate the accuracy and stability of the staggered displacement and the modified staggered pressure methods. The proposed staggered methods were found to be quite accurate when compared with the existing finite element solution.

1. INTRODUCTION

When subjected to earthquake ground motion, the analysis of the dam–reservoir interaction effects is a complex problem. Traditionally, the linear dynamic response of the dam is obtained in the frequency domain. Another approach to determine the linear and nonlinear response of the dam–reservoir system is to approximate the reservoir effects by a number of masses that are added to the dam equation. This is known as the added mass approach. There is evidence that the added mass approximation may not be suitable for problems such as those involving the analysis of cracking in the dam structure (Ghaemian & Ghobarah 1998). The time domain solution of the dam–reservoir interaction problem is of current interest. Such solution is needed for application in fracture mechanics analysis of dams subjected to earthquake ground motion.

The dam–reservoir system can be categorized as a coupled field system in which two physical domains of fluid and structure interact only at their interface. In such a problem, the presence of interaction implies that the time response of both subsystems must be evaluated simultaneously (Felippa & Park 1980). Different approaches to the solution of the coupled field problem exist. Field elimination, simultaneous solution and partitioned solution are the three classes of solutions for the coupled field system. The advantages and disadvantages of each method were addressed by Felippa and Park (1980). The field elimination approach is not feasible in the case of nonlinear problems. The reduced system of equations has high-order derivatives which cause some difficulties in applying the initial conditions. The simultaneous solution is time-consuming and involves many operations,

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especially when a large number of elements is used. This method contains matrices with a large bandwidth and consequently requires a large amount of memory, especially for the cases when the existing matrices are not symmetric. The main disadvantage of the first two classes of solution arises from the difficulties encountered in using available software, while the partitioned solution has the capability of using existing software for each subsystem. The staggered solution is a partitioned solution procedure that can be organized in terms of sequential execution of a single-field analyser.

Most of the physical systems are made of subsystems which interact with each other. These physical systems which are referred to as coupled systems, have been investigated by several researchers. Methods of solution vary, depending on the governing differential equations of the subystems, and may lead to different degrees of accuracy and stability of the solution (Park 1980; Park & Felippa 1980). Coupled problems and their numerical solutions were addressed by Felippa & Park (1980), Park & Felippa (1980, 1984), Zienkiewicz & Taylor (1989), Zienkiewicz (1984), and Zienkiewicz & Chan (1989). Zienkiewicz & Chan (1989) proposed an unconditionally stable method for staggered solution of soil-pore fluid interaction problem. Huang (1995) proposed two unconditionally stable methods for the analysis of soil–pore fluid problem. The methods were named pressure correction method and displacement correction method. Zienkiewicz & Chan (1989) presented an unconditionally stable method for staggered solution procedure for the fluid–structure interaction problem. Their method was proved to be unconditionally stable when no damping term was included in the equations of the fluid and the structure. However, when the damping term is included in the equation of the subsystems, the proposed method may not be unconditionally stable. The problem of solution instability when the damping term is included in the differential equation, was recognized by Wood (1990). Most of the staggered solution applications in the field of fluid-structure interaction were conducted using a method which is not unconditionally stable (Zienkiewicz & Newton 1969; Paul *et al.* 1981).

In this study, two methods of staggered solution procedure are applied to the dam-reservoir interaction problem. Both methods are shown to be unconditionally stable when the two differential equations of the fluid and structure include damping terms. The accuracy of the solution using both of the proposed methods, is investigated. Two different configurations of concrete gravity dams are analysed to illustrate the applicability of the proposed procedure and to compare the solution with available finite element solutions using the added mass approximation for the fluid–structure interaction.

2. THE COUPLED DAM-RESERVOIR PROBLEM

The dam–reservoir interaction is a classic coupled problem, which contains two differential equations of the second order. The equations of the dam structure and the reservoir can be written in the following form:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{f_1\} - [M]\{\ddot{U}_g\} + [Q]\{P\} = \{F_1\} + [Q]\{P\}, \tag{1}$$

$$[G]\{\ddot{P}\} + [C']\{\dot{P}\} + [K']\{P\} = \{F\} - \rho[Q]^{\mathrm{T}}(\{\ddot{U}\} + \{\ddot{U}_g\}) = \{F_2\} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\}, \tag{2}$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices of the structure and $[G]$, $[C']$ and $[K']$ are matrices representing the mass, damping and stiffness of the reservoir, respectively. A detailed definition of the $[G]$, $[C']$ and $[K']$ matrices and vector

$\{F\}$, is presented in the following sections. $[Q]$ is the coupling matrix; $\{f_1\}$ is the vector of body force and hydrostatic force; and $\{P\}$ and $\{U\}$ are the vectors of hydrodynamic pressures and displacements. $\{\ddot{U}_g\}$ is the ground acceleration and ρ is the density of the fluid. The over-dot represents the time derivative.

3. FINITE ELEMENT MODELLING OF THE RESERVOIR

The hydrodynamic pressure distribution in the reservoir is governed by the pressure wave equation. Assuming that water is linearly compressible and neglecting its viscosity, the small-amplitude irrotational motion of water is governed by the two-dimensional wave equation

$$\nabla^2 P(x, y, t) = \frac{1}{V^2} \ddot{P}(x, y, t), \tag{3}$$

where $P(x, y, t)$ is the hydrodynamic pressure in excess of hydrostatic pressure, V is the velocity of pressure wave in water, and x and y are the coordinate axes.

The hydrodynamic pressure in the impounded water, governed by equation (3), is due to the horizontal and the vertical accelerations of the upstream face of the dam, the reservoir bottom as well as the far end of the reservoir in the case of finite reservoir length. The motion of these boundaries is related to the hydrodynamic pressure by the boundary conditions.

For earthquake excitation, the condition at the boundaries of the dam–reservoir, reservoir–foundation and the reservoir-far-end are governed by

$$\frac{\partial P(x, y, t)}{\partial n} = -\rho a_n(x, y, t), \tag{4}$$

where $a_n(x, y, t)$ is the component of acceleration on the boundary along the direction of the inward normal, n. No wave absorption is considered at the boundaries of the reservoir.

For most concrete gravity dams, Eatok Taylor (1981) has shown that free-surface waves are negligible. On this basis, the boundary condition at the free surface is

$$P(x, h, t) = 0, \tag{5}$$

where h is the height of the reservoir.

Using finite element discretization of the fluid domain and the discretized formulation of equation (3), the wave equation can be written in the following matrix form:

$$[G]\{\ddot{P}\} + [H]\{P\} = \{F\}, \tag{6}$$

where $G_{ij} = \Sigma G^e_{ij}$, $H_{ij} = \Sigma H^e_{ij}$ and $F_i = \Sigma f^e_i$.

The coefficients G^e_{ij}, H^e_{ij} and F^e_i for an individual element are determined using the following expressions:

$$G^e_{ij} = \frac{1}{V^2} \int_{A_e} N_i N_j \, \mathrm{d}A, \tag{7}$$

$$H^e_{ij} = \int_{A_e} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) \mathrm{d}A, \tag{8}$$

$$F^e_i = \int_{S_e} N_i \frac{\partial P}{\partial n} \, \mathrm{d}s, \tag{9}$$

where N_i and N_j are the element shape functions, A_e is the element area and, s_e is the prescribed length along the boundary of the elements. In the above formulation, matrices $[H]$ and $[G]$ are constant during the analysis, while the force vector $\{F\}$ and the pressure vector $\{P\}$ and its derivatives are the variable quantities in equation (6).

4. TRUNCATED BOUNDARY AT THE FAR END OF THE RESERVOIR

In order to determine the hydrodynamic pressure on the dam due to horizontal ground motion under the assumption of infinite reservoir, the reservoir must be truncated at a reasonable distance. Sommerfeld boundary condition is a commonly used approach which is based on the assumption that at a far distance from the dam face, the outgoing wave can be considered as a plane wave. Hanna & Humar (1982), Humar & Roufaiel (1983) and Sharan (1986, 1987) used a radiation condition which models the loss of the outgoing wave over a wide range of excitation frequencies. In the present analysis, the Sharan (1986) radiation boundary condition was used. This condition is a suitable one for the time-domain analysis. Other transmitting boundary conditions (Yang *et al.* 1993) are more accurate than Sharan's; however, the simplicity of the selected boundary condition is a major advantage.

The Sharan boundary condition at the far-end truncated boundary can be written as:

$$\frac{\partial P}{\partial n} = -\frac{\pi}{2h}P - \frac{1}{V}\dot{P}. \tag{10}$$

Implementation of the truncated boundary condition in the finite element model, can be done by separating the force vector $\{F\}$ in equation (6) into two components:

$$\{F\} = \{FF_1\} + \{FF_2\}, \tag{11}$$

where $\{\mathrm{FF}_1\}$ is the component of the force due to acceleration at the boundaries of the dam–reservoir and reservoir–foundation while $\{FF_2\}$ is due to truncation at the far boundary.

$$\{FF_2\} = -\frac{\pi}{2h}[D]\{P\} - \frac{1}{V}[D]\{\dot{P}\}, \tag{12}$$

where $D_{ij} = \Sigma D^{e}_{ij}$, and D^{e}_{ij} is defined as

$$D^{e}_{ij} = \int_{l^{e}_{T}} N_i N_j \, \mathrm{d}l_{T}. \tag{13}$$

In equation (13), l^{e}_{T} is the side of the element on the truncated boundary. Substituting equations (11) and (12) into equation (6) results in

$$[G]\{\ddot{P}\} + \frac{1}{V}[D]\{\dot{P}\} + \left([H] + \frac{\pi}{2h}[D]\right)\{P\} = \{FF_1\}. \tag{14}$$

The physical meaning of the boundary condition given by equation (10) is illustrated by examining equation (14). Truncation of the boundary is equivalent to adding dampers and springs to absorb the outgoing waves.

Putting equation (14) in the format of equation (2), leads to

$$[C'] = \frac{1}{V}[D], \quad [K'] = [H] + \frac{\pi}{2h}[D], \quad \{F_2\} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\} = \{FF_1\}. \tag{15}$$

5. TIME-STEPPING SCHEME OF THE COUPLED EQUATIONS

A direct integration scheme is used to find the displacement and hydrodynamic pressure at the end of the time increment $i+1$, given the displacement and hydrodynamic pressure at time i. The Newmark-β method is used for discretization of both equations (implicit–implicit method). In this method $\{\dot{U}\}_{i+1}$, $\{U\}_{i+1}$, $\{\dot{P}\}_{i+1}$ and $\{P\}_{i+1}$ can be written as follows:

$$\{\dot{U}\}_{i+1} = \{\dot{U}\}_{i+1}^{p} + \gamma\Delta t\{\ddot{U}\}_{i+1}, \quad \{\dot{U}\}_{i+1}^{p} = \{\dot{U}\}_i + (1-\gamma)\Delta t\{\ddot{U}\}_i; \tag{16}$$

$$\{U\}_{i+1} = \{U\}_{i+1}^{p} + \beta\Delta t^2\{\ddot{U}\}_{i+1}, \quad \{U\}_{i+1}^{p} = \{U\}_i + \Delta t\{\dot{U}\}_i + (0{\cdot}5-\beta)\Delta t^2\{\ddot{U}\}_i; \tag{17}$$

$$\{\dot{P}\}_{i+1} = \{\dot{P}\}_{i+1}^{p} + \gamma\Delta t\{\ddot{P}\}_{i+1}, \quad \{\dot{P}\}_{i+1}^{p} = \{\dot{P}\}_i + (1-\gamma)\Delta t\{\ddot{P}\}_i; \tag{18}$$

$$\{P\}_{i+1} = \{P\}_{i+1}^{p} + \beta\Delta t^2\{\ddot{P}\}_{i+1}, \quad \{P\}_{i+1}^{p} = \{P\}_i + \Delta t\{\dot{P}\}_i + (0{\cdot}5-\beta)\Delta t^2\{\ddot{P}\}_i; \tag{19}$$

where γ and β are the integration parameters.

The governing field equations at time $i+1$ can be written as follows:

$$[M]\{\ddot{U}\}_{i+1} + [C]\{\dot{U}\}_{i+1} + [K]\{U\}_{i+1} = \{F_1\}_{i+1} + [Q]\{P\}_{i+1}, \tag{20}$$

$$[G]\{\ddot{P}\}_{i+1} + [C']\{\dot{P}\}_{i+1} + [K']\{P\}_{i+1} = \{F_2\}_{i+1} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\}_{i+1}. \tag{21}$$

The coupled field equations (20) and (21) can be solved using the staggered solution scheme. The procedure can be started by guessing $\{P\}_{i+1}$ in equation (20) to solve for $\{U\}_{i+1}$ and its derivatives. Then, equation (21) can be solved to find $\{P\}_{i+1}$. This method cannot guarantee the unconditional stability of the solution. Similarly, guessing $\{\ddot{U}\}_{i+1}$ at first to calculate $\{P\}_{i+1}$ from equation (21), and then calculating $\{U\}_{i+1}$ from equation (20) cannot provide an unconditionally stable procedure.

In the following section, two methods of staggered solution are proposed and are shown to be unconditionally stable.

6. STAGGERED DISPLACEMENT METHOD

In this method, equation (20) can be approximated as follows:

$$[M]\{\ddot{U}\}_{i+1}^{*} = \{F_1\}_{i+1} + [Q]\{P\}_{i+1}^{p} - [C]\{\dot{U}\}_{i+1}^{p} - [K]\{U\}_{i+1}^{p}. \tag{22}$$

Introducing equations (16), (17) and (19) into equation (20) and substituting into equation (22), the following equation is obtained:

$$[M]\{\ddot{U}\}_{i+1} = [M]\{\ddot{U}\}_{i+1}^{*} + \beta\Delta t^2[Q]\{\ddot{P}\}_{i+1} - \gamma\Delta t[C]\{\ddot{U}\}_{i+1} - \beta\Delta t^2[K]\{\ddot{U}\}_{i+1}. \tag{23}$$

The lumped mass assumption will result in a diagonal mass matrix. However, in general, the off-diagonal terms of the matrices $[C]$ and $[K]$ are not zero. In the seismic analysis of a typical concrete gravity dam, the parameters β and γ are taken as 0·25 and 0·5, respectively, and the time step Δt is normally taken less than 0·02 s. The numerical value of each of the two terms $\beta\Delta t^2[K]$ and $\gamma\Delta t[C]$ are much smaller than $[M]$. By neglecting these two terms, equation (23) is approximated as

$$[M]\{\ddot{U}\}_{i+1} = [M]\{\ddot{U}\}^*_{i+1} + \beta\Delta t^2[Q]\{\ddot{P}\}_{i+1}. \tag{24}$$

Substituting equation (24) into equation (21), yields

$$([G] + \rho\beta\Delta t^2[Q]^{\mathrm{T}}[M]^{-1}[Q])\{\ddot{P}\}_{i+1} + [C']\{\dot{P}\}_{i+1} + [K']\{P\}_{i+1} = \{F_2\}_{i+1} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\}^*_{i+1}. \tag{25}$$

In equation (25), the right-hand-side terms are known; thus, $\{P\}_{i+1}$ can be obtained. In order to correct the approximation made in equation (24), $\{P\}_{i+1}$ can be substituted in equation (20) to calculate $\{U\}_{i+1}$ and its derivatives.

Therefore, the procedure of the staggered displacement method can be summarized by the following steps: (i) solving equation (22) to calculate $\{\ddot{U}\}^*_{i+1}$; (ii) substituting $\{\ddot{U}\}^*_{i+1}$ into equation (25) to calculate $\{P\}_{i+1}$; (iii) substituting $\{P\}_{i+1}$ into equation (20) to calculate $\{U\}_{i+1}$ and its derivatives.

7. STABILITY OF THE STAGGERED DISPLACEMENT METHOD

In an unconditionally stable solution method, an instability can be attributed to an actual instability of the structure. While in a conditionally stable method, the instability may be due to either numerical or structural instability. To show that the described method of staggered displacement is unconditionally stable, consider a modally decomposed system with scalar values. In such a system, the displacement and the pressure must not grow. Thus for $|\mu| < 1$, we have

$$\{U\}_{i+1} = \mu\{U\}_i, \quad \{\dot{U}\}_{i+1} = \mu\{\dot{U}\}_i, \quad \{\ddot{U}\}_{i+1} = \mu\{\ddot{U}\}_i; \tag{26}$$

$$\{P\}_{i+1} = \mu\{P\}_i, \quad \{\dot{P}\}_{i+1} = \mu\{\dot{P}\}_i, \quad \{\ddot{P}\}_{i+1} = \mu\{\ddot{P}\}_i. \tag{27}$$

Using the transformation $\mu = (1 + s)/(1 - s)$, the condition for stability requires that the real part of s be negative, $\mathrm{Re}(s) \leq 0$, and that the Routh–Hurwitz criterion (Wood 1990; Zienkiewicz and Taylor 1989) apply. For $\beta = 0{\cdot}25$ and $\gamma = 0{\cdot}5$, equations (16)–(19) become

$$\{\ddot{U}\}_{i+1} = \frac{4s^2}{\Delta t^2}\{U\}_{i+1}, \quad \{\dot{U}\}_{i+1} = \frac{2s}{\Delta t}\{U\}_{i+1};$$

$$\{\dot{U}\}^p_{i+1} = \frac{2s - s^2}{\Delta t}\{U\}_{i+1}, \quad \{U\}^p_{i+1} = (1 - s^2)\{U\}_{i+1}; \tag{28}$$

$$\{\ddot{P}\}_{i+1} = \frac{4s^2}{\Delta t^2}\{P\}_{i+1}, \quad \{\dot{P}\}_{i+1} = \frac{2s}{\Delta t}\{P\}_{i+1};$$

$$\{\dot{P}\}^p_{i+1} = \frac{2s - s^2}{\Delta t}\{P\}_{i+1}, \quad \{P\}^p_{i+1} = (1 - s^2)\{P\}_{i+1}. \tag{29}$$

Rewriting equation (20) without the force terms, gives

$$[M]\{\ddot{U}\}_{i+1} + [C]\{\dot{U}\}_{i+1} + [K]\{U\}_{i+1} - [Q]\{P\}_{i+1} = 0. \tag{30}$$

Combining equations (23) and (25) and substituting them into equation (21) without the force term, gives

$$[G]\{\ddot{P}\}_{i+1} + [C']\{\dot{P}\}_{i+1} + [K']\{P\}_{i+1} + \rho[Q]^{\mathrm{T}}[M]^{-1}([M] + \gamma\Delta t[C] + \beta\Delta t^2[K])\{\ddot{U}\}_{i+1} = 0. \tag{31}$$

The modally decomposed system is represented by a single-degree-of-freedom equation. The single-degree-of-freedom equivalent of equations (30) and (31) will be obtained by substituting the mass, damping and stiffness values m, c and k, instead of $[M]$, $[C]$ and $[K]$ in equation (30), and g, c' and k' instead of $[G]$, $[C']$ and $[K']$ in equation (31). The coupling matrix $[Q]$ would be represented by scalar quantity q. The characteristic equation of the coupled field is obtained by substituting equations (28) and (29) into equations (30) and (31) as follows:

$$\begin{vmatrix} m\dfrac{4s^2}{\Delta t^2} + c\dfrac{2s}{\Delta t} + k & -q \\ \dfrac{\rho q}{m}\left(m + \dfrac{\Delta t}{2}c + \dfrac{\Delta t^2}{4}k\right)\dfrac{4s^2}{\Delta t^2} & g\dfrac{4s^2}{\Delta t^2} + c'\dfrac{2s}{\Delta t} + k' \end{vmatrix} = 0, \tag{32}$$

or

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0, \tag{33}$$

where

$$a_0 = \frac{16mg}{\Delta t^4}, \quad a_1 = \frac{8mc'}{\Delta t^3} + \frac{8gc}{\Delta t^3},$$

$$a_2 = \frac{4mk'}{\Delta t^2} + \frac{4cc'}{\Delta t^2} + \frac{4gk}{\Delta t^2} + \frac{4\rho q^2}{\Delta t^2} + \frac{2\rho q^2 c}{m\Delta t} + \frac{\rho q^2 k}{m},$$

$$a_3 = \frac{2c'k}{\Delta t} + \frac{2ck'}{\Delta t}, \quad a_4 = kk'. \tag{34}$$

The Routh–Hurwitz conditions for stability are

$$a_0 > 0, \quad a_1, a_2, a_3, a_4 \geq 0, \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0. \tag{35}$$

For the structural system of dam and reservoir, m, c, k, g, c' and k' are positive quantities. Therefore, a_0, a_1, a_2, a_3 and a_4 are always positive. The values of the two determinants in equation (35) are given as:

$$a_1 a_2 - a_3 a_0 = \frac{32}{\Delta t^5}m^2 c'k' + \frac{32}{\Delta t^5}mc'^2 c + \frac{32\rho}{\Delta t^5}mc'q^2 + \frac{16\rho}{\Delta t^4}c'q^2 c + \frac{8\rho}{\Delta t^3}c'q^2 k + \frac{32}{\Delta t^5}gc^2 c' + \frac{32}{\Delta t^5}g^2 ck + \frac{32\rho}{\Delta t^5}gcq^2 + \frac{16\rho}{\Delta t^4}gc^2 q^2 + \frac{8\rho}{\Delta t^3}gcq^2 k, \tag{36}$$

$$
\begin{aligned}
a_1a_2a_3 - a_1^2a_4 - a_3^2a_0 = {} & \frac{64}{\Delta t^6}(mk'\sqrt{cc'} - gk\sqrt{cc'})^2 + \frac{64}{\Delta t^6}mc'^3ck + \frac{64}{\Delta t^6}mc'^2c^2k' \\
& + \frac{64\rho}{\Delta t^6}mc'^2q^2k + \frac{64\rho}{\Delta t^6}mc'q^2ck' + \frac{32\rho}{\Delta t^5}c'^2q^2ck \\
& + \frac{32\rho}{\Delta t^5}c'q^2c^2k' + \frac{16\rho}{\Delta t^4}c'^2q^2k^2 + \frac{16\rho}{\Delta t^4}c'q^2kck' \\
& + \frac{64}{\Delta t^6}gc^2c'^2k + \frac{64}{\Delta t^6}gc^3c'k' + \frac{64\rho}{\Delta t^6}gcq^2c'k \\
& + \frac{64\rho}{\Delta t^6}gc^2q^2k' + \frac{32\rho}{\Delta t^5}gc^2q^2c'k + \frac{32\rho}{\Delta t^5}gc^3q^2k' \\
& + \frac{16\rho}{\Delta t^4}gcq^2k^2c' + \frac{16\rho}{\Delta t^4}gc^2q^2kk'. \qquad (37)
\end{aligned}
$$

All the terms in equations (36) and (37) are positive. Recalling the condition of stability (35), then the method of staggered displacement is unconditionally stable.

8. STAGGERED PRESSURE METHOD

In this method, the vector $\{\ddot{P}\}^*_{i+1}$ is defined using equations (18), (19) and (21) as

$$[G]\{\ddot{P}\}^*_{i+1} = \{F_2\}_{i+1} - [C']\{\dot{P}\}^p_{i+1} - [K']\{P\}^p_{i+1}. \qquad (38)$$

Substituting equation (38) into equation (21), leads to

$$[G]\{\ddot{P}\}_{i+1} = [G]\{\ddot{P}\}^*_{i+1} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\}_{i+1} - \gamma\Delta t[C']\{\ddot{P}\}_{i+1} - \beta\Delta t^2[K']\{\ddot{P}\}_{i+1}, \qquad (39)$$

or

$$([G] + \beta\Delta t^2[K'] + \gamma\Delta t[C'])\ \{\ddot{P}\}_{i+1} = [G]\{\ddot{P}\}^*_{i+1} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\}_{i+1}. \qquad (40)$$

Substituting equation (40) into equation (20) with $[J] = [G] + \beta\Delta t^2[K'] + \gamma\Delta t[C']$, gives

$$
\begin{aligned}
([M] + \rho\beta\Delta t^2[Q][J]^{-1}[Q]^{\mathrm{T}})\{\ddot{U}\}_{i+1} + [C]\{\dot{U}\}_{i+1} + [K]\{U\}_{i+1} \\
= \{F_1\}_{i+1} + [Q]\{P\}^p_{i+1} + \beta\Delta t^2[J]^{-1}[G]\{\ddot{P}\}^*_{i+1}). \qquad (41)
\end{aligned}
$$

Using equation (41), the variable $\{\ddot{U}\}_{i+1}$ can be calculated. Substituting $\{\ddot{U}\}_{i+1}$ into equation (20) gives $\{P\}_{i+1}$ and its derivatives.

Therefore, the procedure of staggered pressure method can be summarized by the following steps: (i) solving equation (38) to calculate $\{\ddot{P}\}^*_{i+1}$; (ii) substituting $\{\ddot{P}\}^*_{i+1}$ into equation (41) to calculate $\{\ddot{U}\}_{i+1}$; (iii) substituting $\{\ddot{U}\}_{i+1}$ into equation (40) to calculate $\{P\}_{i+1}$ and its derivatives.

9. STABILITY OF THE STAGGERED PRESSURE METHOD

For a stability check, a similar procedure as that used in the displacement method can be applied. Rewriting equations (20) and (21) without the force terms, yields

$$[M]\{\ddot{U}\}_{i+1} + [C]\{\dot{U}\}_{i+1} + [K]\{U\}_{i+1} - [Q]\{P\}_{i+1} = 0, \qquad (42)$$

$$[G]\{\ddot{P}\}_{i+1} + [C']\{\dot{P}\}_{i+1} + [K']\{\ddot{P}\}_{i+1} + \rho[Q]^{\mathrm{T}}\{\ddot{U}\}_{i+1} = 0. \qquad (43)$$

The characteristic equation of the coupled field for a modally decomposed system with scalar values, can be obtained by substituting equations (28) and (29) into equations (42) and (43):

$$\begin{vmatrix} m\dfrac{4s^2}{\Delta t^2} + c\dfrac{2s}{\Delta t} + k & -q \\ \rho q\dfrac{4s^2}{\Delta t^2} & g\dfrac{4s^2}{\Delta t^2} + c'\dfrac{2s}{\Delta t} + k' \end{vmatrix} = 0, \tag{44}$$

or

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0, \tag{45}$$

where

$$\begin{aligned} a_0 &= \frac{16mg}{\Delta t^4}, \quad a_1 = \frac{8mc'}{\Delta t^3} + \frac{8gc}{\Delta t^3}, \\ a_2 &= \frac{4mk'}{\Delta t^2} + \frac{4cc'}{\Delta t^2} + \frac{4gk}{\Delta t^2} + \frac{4\rho q^2}{\Delta t^2}, \\ a_3 &= \frac{2c'k}{\Delta t} + \frac{2ck'}{\Delta t}, \quad a_4 = kk'. \end{aligned} \tag{46}$$

The coefficients of the polynomial are all positive. The determinants in the Routh–Hurwitz conditions, equations (35), give

$$a_1 a_2 - a_3 a_0 = \frac{32}{\Delta t^5} m^2 c' k' + \frac{32}{\Delta t^5} m c'^2 c + \frac{32\rho}{\Delta t^5} m c' q^2 + \frac{32}{\Delta t^5} g c^2 c' + \frac{32}{\Delta t^5} g^2 ck + \frac{32\rho}{\Delta t^5} gcq^2, \tag{47}$$

$$a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 = \frac{64}{\Delta t^6}(mk'\sqrt{cc'} - gk\sqrt{cc'})^2 + \frac{64}{\Delta t^6} m c'^3 ck + \frac{64}{\Delta t^6} m c'^2 c^2 k'$$

$$+ \frac{64\rho}{\Delta t^6} m c'^2 q^2 k + \frac{64\rho}{\Delta t^6} m c' q^2 c k' + \frac{64}{\Delta t^6} g c^3 c' k' + \frac{64}{\Delta t^6} g c^2 c'^2 k + \frac{64\rho}{\Delta t^6} g c^2 q^2 k' + \frac{64\rho}{\Delta t^6} g c q^2 c' k. \tag{48}$$

These terms are all positive. Therefore, given the stability conditions (35), the method of staggered pressure is unconditionally stable.

10. MODIFIED STAGGERED PRESSURE METHOD

Most of the available nonlinear solutions assume a diagonal mass matrix for the purpose of analysis. The staggered displacement method is the most suitable coupled field problem solution procedure for the case of nonlinear analysis. In the case of the staggered pressure method, some difficulties may arise due to the added mass terms in equation (41) which change the mass matrix from a diagonal to a full matrix. For this reason, a modification to the staggered pressure method is proposed to make it applicable to nonlinear analysis.

The staggered pressure method is modified by rewriting equation (41) in the following approximate form:

$$\begin{aligned} &[M]\{\ddot{U}\}_{i+1} + [C]\{\dot{U}\}_{i+1} + [K]\{U\}_{i+1} \\ &= \{F_1\}_{i+1} + [Q](\{P\}_{i+1}^{p} + \beta\Delta t^2[J]^{-1}([G]\{\ddot{P}\}_{i+1}^{*} - \rho[Q]^{\mathrm{T}}\{\ddot{U}\}_i)). \end{aligned} \tag{49}$$

Therefore, the procedure of the modified staggered pressure method can be summarized by the following steps: (i) solving equation (38) to calculate $\{\ddot{P}\}^*_{i+1}$; (ii) substituting $\{\ddot{P}\}^*_{i+1}$ into equation (49) to calculate $\{\ddot{U}\}_{i+1}$; (iii) substituting $\{\ddot{U}\}_{i+1}$ into equation (40) to calculate $\{P\}_{i+1}$ and its derivatives.

The modified staggered pressure method does not guarantee unconditional stability of the solution.

11. ACCURACY OF THE SOLUTION SCHEME

The accuracy of the staggered solution scheme can be improved by increasing the number of iterations and/or by decreasing the time step. Increasing the number of iterations of the staggering scheme is a time-consuming process. The accuracy of the proposed methods is based on the selection of the appropriate time step. In all of the following analyses, no iterations have been made for the purpose of improving accuracy. The staggered displacement method and the modified staggered pressure method are compared with the finite element solution of example problems for the purpose of evaluating the accuracy of the analysis.

12. NUMERICAL RESULTS

Two cases of concrete gravity dams with different reservoir levels were analysed to demonstrate the applicability and accuracy of the proposed methods. The modulus of elasticity, unit weight and Poisson's ratio of concrete were taken as 3430 MPa, 2400 kg/m^3 and 0·2, respectively. The selected dam–reservoir systems for the two cases of numerical examples are shown in Figures 1 and 2. In the first example, a full reservoir is considered and the structure has a fundamental frequency of 6·837 rad/s. The second example has

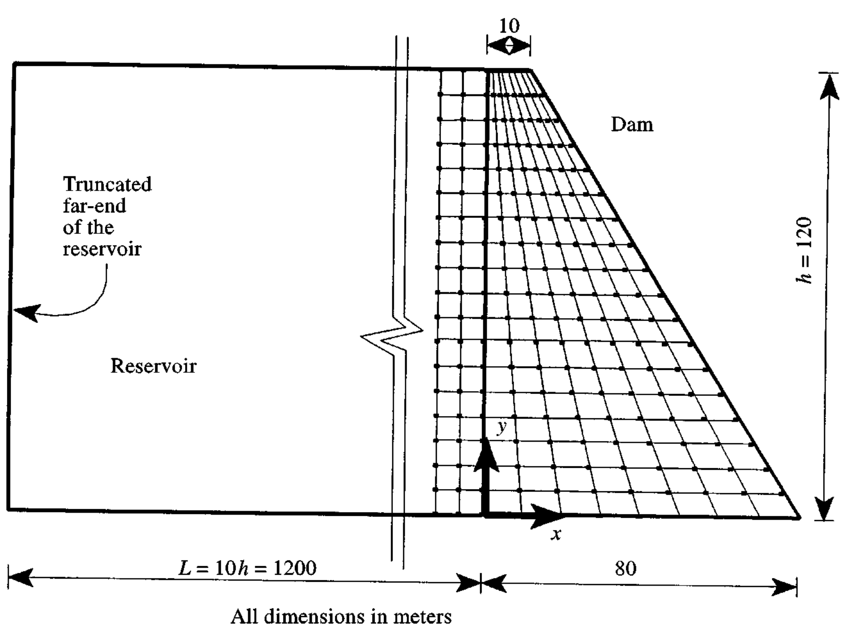

Figure 1. Finite element model of the dam–reservoir system in example 1.

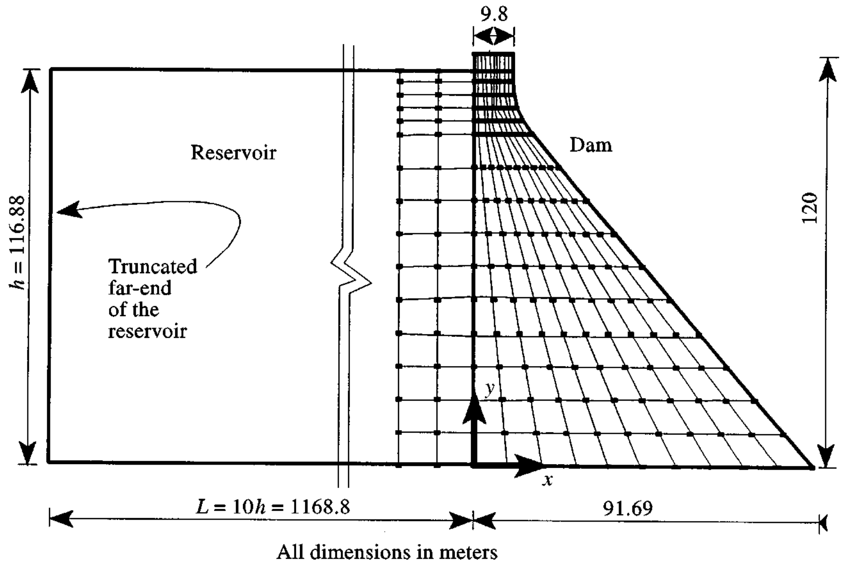

Figure 2. Finite element model of the dam–reservoir system in example 2.

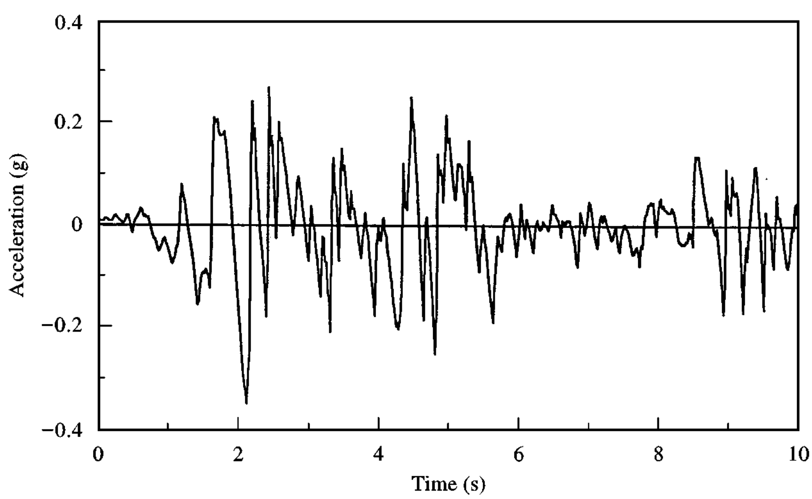

Figure 3. First 10 s of the horizontal component of the 18 May 1940 Imperial Valley earthquake, El Centro record.

a typical configuration of a concrete gravity dam of fundamental frequency of 7·57 rad/s with a partially filled reservoir.

Figure 3 shows 10 s of the horizontal S00E component of the 18 May 1940 Imperial Valley earthquake, El Centro site record, which is selected for the purpose of dynamic analysis. The ground motion has a peak acceleration of 0·348g. The values of the integration parameters in the Newmark-β method were taken as $\beta = 0{\cdot}25$ and $\gamma = 0{\cdot}5$. The velocity of pressure waves in water was taken as 1438.66 m/s.

In the absence of reliable actual measurements of dam crest displacement to a known ground motion or experimental data, the staggered solution is compared with the well-established frequency domain analysis. Results of the analysis using the staggered displacement and modified pressure methods are compared with the dynamic analysis using EAGD-84 (Fenves & Chopra 1984) program which assumes infinite reservoir length. In the finite element formulation of the reservoir, the Sharan boundary condition (Sharan 1986), which truncates the reservoir, was applied at a distance from the dam equal to 10 times the dam height.

The EAGD-84 is a computer code in the frequency domain for the linear analysis of the dam–reservoir interaction which gives the steady state response of the system. The results presented using the staggered methods are obtained from the time-domain analysis, which include the steady-state and transient responses of the system. In the case of a typical concrete gravity dam, the transient response is negligible. Four-node isoparametric elements were used to represent the finite elements of the structure and the fluid domains. Stiffness proportional damping (Rayleigh damping) is used. The modified staggered pressure method is used instead of the staggered pressure method, and the results are compared with those obtained using the staggered displacement analysis procedure.

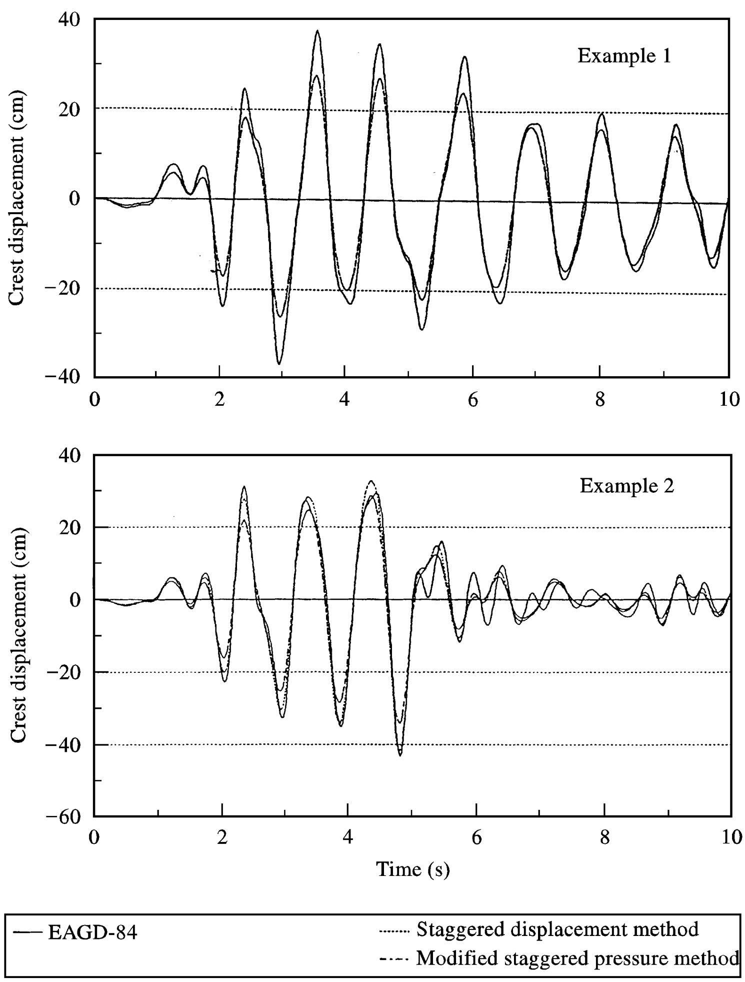

Figure 4. Dam crest displacement versus time.

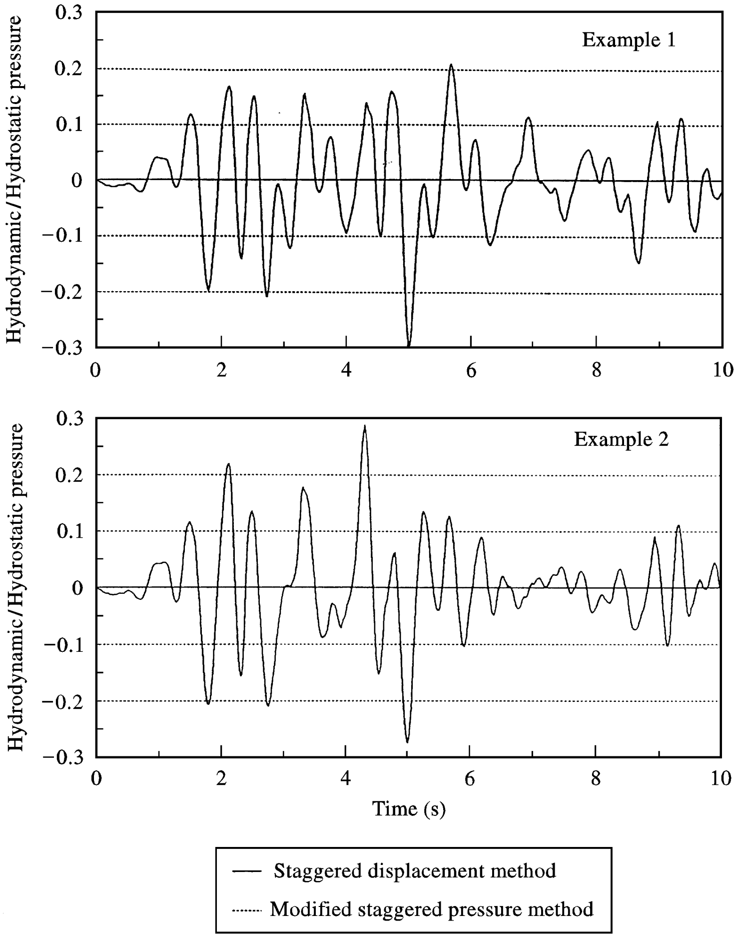

Figure 5. Hydrodynamic pressure time history near the bottom of the dam.

Figure 4 shows the results of the analysis for the dam crest displacement of the two dam examples. For a time step dt = 0·001 s, excellent agreement is found between the response obtained from the two proposed methods and the EAGD-84 solution.

The hydrodynamic pressure time histories on the upstream face near the bottom of the dams in the two examples are shown in Figure 5. Results of the staggered displacement method and the modified staggered pressure method coincide.

Figures 6 and 7 show the results of the analysis obtained using different time steps. The figures show that the staggered displacement method is accurate even for the large time step of dt = 0·02 s. In the case of the modified staggered pressure method using time step smaller than 0·004 s, good results are obtained. Using time steps larger than 0·004 s in the modified pressure method leads to numerical instability of the solution.

The fundamental periods of the dams in the examples 1 and 2 are 0·92 and 0·83 s, respectively. Normalizing the time step to the fundamental structural period will result in a ratio of the same order of magnitude as the time step itself. The time step used in the analysis is small in comparison with the fundamental period of the structure. However, in the analysis of concrete dams, at least the first three modes of vibration may contribute significantly to the response.

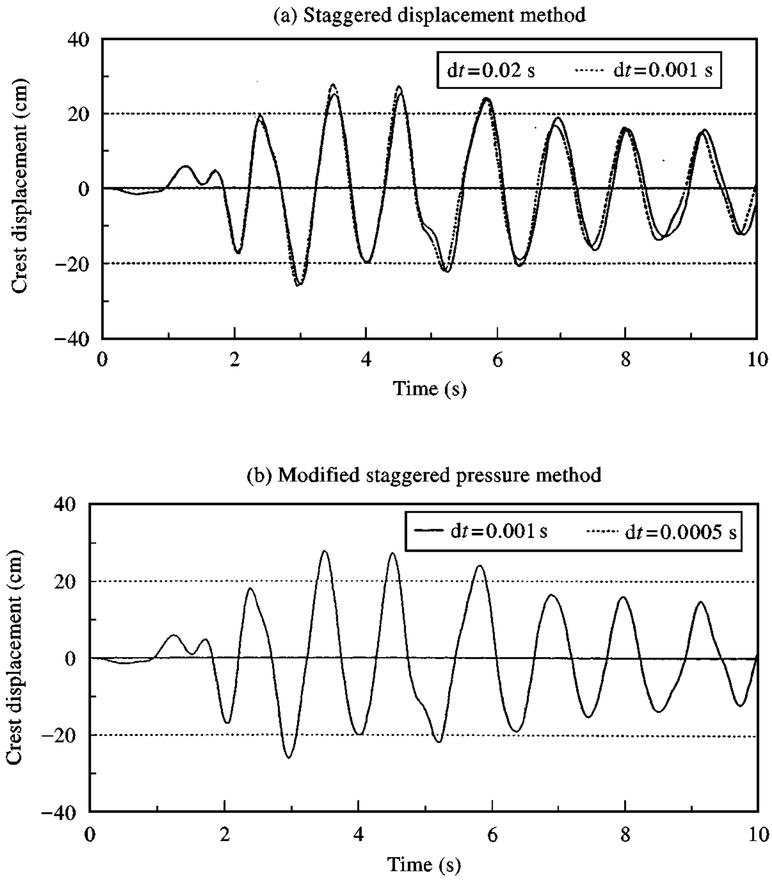

Figure 6. Accuracy of the proposed method with different time steps in example 1.

13. CONCLUSIONS

Two methods of staggered solution procedure for the dam–reservoir coupled system are introduced. The staggered displacement method is based on the approximation of the displacement in the equation of motion of the structure. The staggered pressure method is based on the approximation of the pressure in the fluid equation of motion. Both methods are proved to be unconditionally stable when the two differential equations of the fluid and structure include damping terms. The displacement and modified staggered pressure methods are suitable for nonlinear analysis. Two cases of concrete gravity dams are analysed to investigate the accuracy and stability of the staggered displacement method. The method is found to be accurate when compared with the frequency-domain finite element solution. No instability is observed in the analysis in the case of the displacement method. However, in the case of the modified staggered pressure method, numerical instability is observed for large time steps. It is concluded that the displacement method gives stable solution with accurate results even for a large time step. The solution procedure is found to be less time-consuming than the frequency-domain solution. The modified staggered pressure method is applicable in problems which are modelled by full mass matrices.

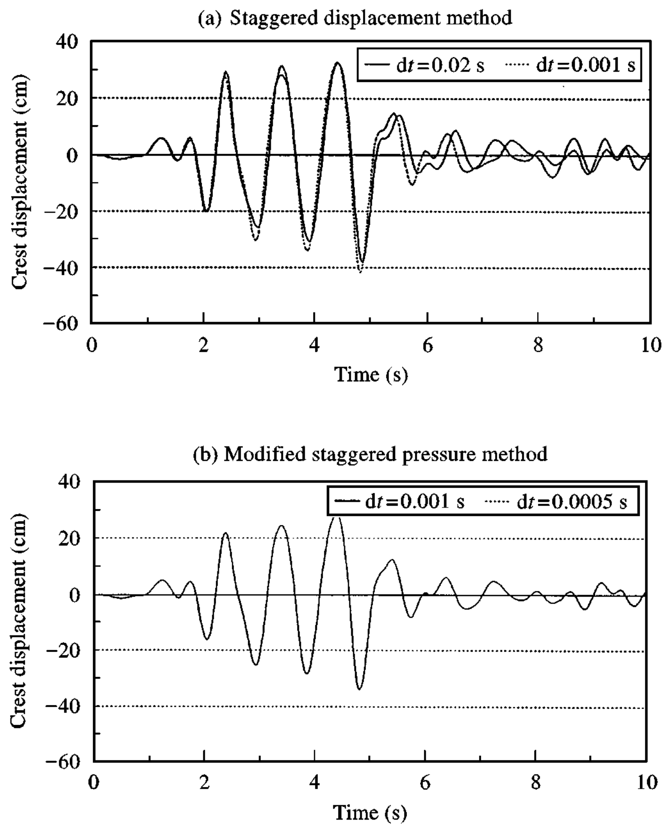

Figure 7. Accuracy of the proposed method with different time steps in example 2.

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